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COVERING SIMPLY CONNECTED REGIONS BY RECTANGLES

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We prove that the ratio of the minimum number of rectangles covering a simply connected board (polyomino) B and the maximum number of points in B no two of which are contained in a common rectangle is less than 2.

A *board (polyomino)* is a finite set of unit squares lying in the plane whose corners have integer coordinates. In other words, a board is a lattice polygon with vertical and horizontal sides. A *rectangle* is a subset of the board whose union is rectangular. A *cover* of a board B is a collection of rectangles whose union is equal to B. (That is, the rectangles of a cover may overlap but they must be contained in B.) An *antirectangle* in B is a set of squares in B no two of which are contained in a common rectangle. It is obvious that any cover has to contain at least as many rectangles as any antirectangle has squares. Thus if $\theta(B)$ is the minimum number of rectangles in a cover of B and $\alpha(B)$ is the maximum number of squares in an antirectangle of B then $\alpha(B)\leqq\theta(B)$. V. Chvátal originally conjectured that $\alpha(B)=\theta(B)$ holds for any finite board B. In general, this is false. First E. Szemerédi [4] found a counterexample with a "hole" (Figure 2), then F.R.K. Chung [2] found the simply connected counterexample in Figure 1. Then S. Chaiken, D. J. Kleitman, M. Saks and J. Shearer [1] proved a weakened version of the conjecture that equality does hold if B is horizontally and vertically convex i.e. whenever two squares in B are on the same horizontal or vertical line, all squares between them are in B. Recently, it was proved in [3] that $\alpha(B)=\theta(B)$ holds for any vertically convex board B. Considering Chung's counterexample in Figure 1, this is the most general possible version of the conjecture in some sense.

As it is written in [1], P. Erdős asked if θ/α is bounded and the answer is not known. Chung's example has $\theta/\alpha=8/7$. The most that Chaiken et al. achieved for θ/α is $21/17-\varepsilon$ for any sufficiently small ε. Here we prove the following

Theorem. $\theta(B)\leqq 2\alpha(B)-1$ *for any simply connected board* B.

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Proof. Let B be a simply connected board. Consider the intersections of the (closed) board with the horizontal lines. Every intersection consists of some disjoint intervals. Take the intervals containing a segment of the boundary of B. Let $I_1, I_2, \ldots, I_m$ denote these (maximal) intervals. Now we prove

$$\alpha(B) \geqq m/2 \tag{1}$$

and

$$\theta(B) \leqq m-1 \tag{2}$$

which yield the statement of Theorem.

First we prove that $\alpha(B) \geqq m/2$. By the definition of the intervals $I_1, I_2, \ldots, I_m$, there exists a unit square S_i in B such that the boundary of S_i and the intersection of I_i and the boundary of B have a common segment for $i=1, 2, \ldots, m$. These unit squares S_i are either above or under the intervals I_i, so without loss of generality, we may suppose that for $k \geqq m/2$ indices, e.g. for $i=1, 2, \ldots, k$, S_i is above I_i. Now we prove that the unit squares $S_1, S_2, \ldots, S_k$ constitute an antirectangle.

If S_i and S_j are on different levels (i.e. in different rows), e.g. S_i is on a higher level than S_j then S_i and S_j cannot belong to the same rectangle because such a rectangle would have to contain the lower neighbouring unit square of S_i that does not belong to B by the definition of S_i.

If S_i and S_j $(i \neq j)$ are on the same level then the segment between the lower sides of the unit squares S_i and S_j contains a point P not belonging to B because the lower sides of S_i and S_j are segments of I_i and I_j, respectively and I_i and I_j are maximal segments belonging to B by definition. Then the unit square containing P in the row of S_i and S_j does not belong to B and so S_i and S_j cannot belong to the same rectangle. This completes the proof of inequality (1).

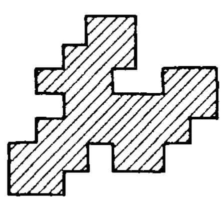

Fig. 1

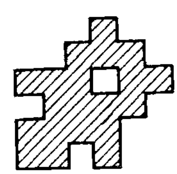

Fig. 2

Now we prove that $\theta(B) \leqq m-1$. Let us consider the intervals $I_1, I_2, \ldots, I_m$. Let us connect two intervals by a vertical segment if we can do it without crossing any other interval and without leaving the board B. Thus we obtain an arrangement A of vertical and horizontal segments in B without crossing. We prove that this arrangement does not contain any circuit (polygon line). Suppose that it is not the case. The angles of P are of 90 or 270 degrees. Suppose that there is an angle of 270 degrees. The horizontal arm of this angle is a subsegment S_0 of a horizontal segment S of the arrangement A. The interior of P belongs to the board B by the simple connectivity of B, and so the subsegment $S-S_0$ intersects the boundary of P by the maximality of S, contradicting the fact that the segments of the arrangement A do not intersect each other. Thus the angles of P are of 90 degrees, P is a rectangle and the horizontal sides of P are connected by at least two vertical segments, a contradiction.

Then consider the graph G with vertex-set $V(G)=\{1, 2, ..., m\}$ and edge-set $E(G)=\{ij\colon I_i$ and I_j are connected by a vertical segment$\}$. Now G does not contain any circuit because it would correspond to a polygon P in the arrangement above. Thus the number $e=|E(G)|$ of the connecting vertical segments is at most $m-1$. Let $T_1, T_2, ..., T_e$ denote these segments. Consider the maximal rectangles R_i containing S_i such that R_i is contained in B and that the orthogonal projection of R_i on the line of S_i is S_i for $i=1, 2, ..., e$.

We prove that the rectangles $R_1, R_2, ..., R_e$ constitute a cover of B. Let X be an arbitrary point of B. Consider the vertical line L through X. Let I_a and L_u denote the intervals intersected by L first above and under X, respectively. Then I_a and I_u are connected by a vertical segment $T_i\in\{T_1, ..., T_e\}$ but they could have been connected by a segment T of L. Then T_i. T and the segments of I_a and I_u between T_i and T are in B, so by the simple connectivity of B, the whole rectangle bounded by these four segments is contained by B. But then R_i covers X and so we have proved that the rectangles $R_1, ..., R_e$ constitute a cover of B.

So we have proved that $\theta(B)\leqq e\leqq m-1$ and the proof of the Theorem is complete. ∎

References

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